In this post, we will continue our exposition of the CCAPM model, by first reformulating the pricing equation in terms of asset returns, and then discuss how risk affects returns. Returns are a useful measure; they are often used in empirical work because they are typically stationary in the statistical sense, i.e., they do not have trends and one can meaningfully take an average.

Note: In what follows, all expectations, covariances, and variances, are taken with respect to the information available to investors up to time t.

1. Returns Formulation

First, we recall the basic pricing equation from our first post on the CCAPM:

$$p_t^i = E(m_{t+1}x_{t+1}^i) \tag{1}$$

where $m_{t+1} := \beta \frac{u'(c_{t+1})}{u'(c_t)}$. Now, we define the **(gross) return** of asset i, R_{t+1}^i, as

$$R_{t+1}^i := \frac{x_{t+1}^i}{p_t}$$

In other words, the return of an asset is the amount gained from investing in the asset relative to the amount paid for the asset. Rearranging (1), to express it in terms of returns, we get

$$1 = E(m_{t+1}R_{t+1}) \tag{2}$$

Now, we consider an asset which has a constant return. We call this asset a risk-free asset and it's return, (gross) **risk free interest rate** R^f at time t. Since R^f is a constant, we can take it out of the expectation and rearrange (2), to obtain

$$R^f = \frac{1}{E(m_{t+1})} \tag{3}$$

It is common to study such equity strategies in which one short sells one portfolio or stock and invests the proceeds in another stock or portfolio, resulting in an **excess return** $R_t^e := R_t - R^f$. It is sometimes called a **zero cost portfolio**.

Though returns are a useful empirical quantity, it is important to note that not every asset can be reduced to a return. For instance, if you borrow a dollar at R^f and invest it in another asset with return R_t, you pay no money out of pocket today, but get the payoff $x_{t+1}^e = R_{t+1} - R^f$ tomorrow. This is a **zero price payoff**, which is a bet in which the value of the value of the chance of winning exactly balances the value of the chance of losing:

$$\begin{aligned} p_t &= E(m_{t+1}x_{t+1}^e) = E(m_{t+1}(R_{t+1} - R^f)) \\ &= E(m_{t+1}R_{t+1}) - E(m_{t+1}R^f) \\ &= 1 - 1 = 0 \end{aligned}$$

2. Risk-Free Interest Rates

Now, to think about the economics behind real interest rates, let us assume the power utility function $u(c_t) = (1-\gamma)c_t^{1-\gamma}$ and assume that there is only a single state in our state space (no uncertainty). Then, the risk free rate will be

$$R^f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} \tag{4}$$

From (4) we can see the following effects:

1. R^f is high when people are impatient (when β is low). In other words, if people highly prefer to consume now as opposed to later, it takes a high interest rate to convince them to save.
2. R^f is high when consumption growth c_{t+1}/c_t is high. Intuitively, in times of high interest rates, it pays investors to consume less now, invest more, and consume more in the future. Thus, high interest rates lower the level of consumption today, while raising its growth rate from today to tomorrow.
3. Real interest rates are more sensitive to consumption growth if the power parameter γ is large. If utility is highly curved, the investor cares more about maintaining a consumption profile that is smooth over time, and is less willing to rearrange consumption over time in response to interest rate incentives. Thus it takes a larger interest rate to induce him to invest in the riskless asset for a given consumption growth.

Many standard forms of uncertainty (e.g. log normal distributed consumption growth) maintain the above characteristics, but additionally capture the notion of precautionary savings. When consumption is more volatile, people are more worried about low consumption states than they are pleased by the high consumption states. Thus, people want to save more, driving down interest rates.

3. Risk Corrections

Using the definition of covariance,

$$\text{cov}(m_{t+1}, x_{t+1}^i) := E(m_{t+1}x_{t+1}^i) - E(m_{t+1})E(x_{t+1}^i)$$

and our expression for the risk free rate, $R^f = 1/E(m_{t+1})$, we can express p_t^i as:

$$\begin{aligned} p_t^i &= E(m_{t+1})E(x_{t+1}^i) + \text{cov}(m_{t+1}, x_{t+1}^i) \\ &= \frac{E(x_{t+1}^i)}{R^f} + \text{cov}(m_{t+1}, x_{t+1}^i) \\ &= \frac{E(x_{t+1}^i)}{R^f} + \frac{\text{cov}(\beta u'(c_{t+1}), x^i)}{u'(c_t)} \end{aligned} \tag{5}$$

- For a moment, let's assume that agents are risk neutral ($u(c) = ac + b$ where $a, b \in \mathbb{R}_+$). In this case, we note that $\text{cov}(m_{t+1}, x_{t+1}^i) = 0$, so we will be left with the first term in (5). That term is known as the standard discounted present value formula. It is the asset's price in a risk neutral world.

- The second term, $\text{cov}(\beta u'(c_{t+1}), x^i)/u'(c_t) = \text{cov}(m_{t+1}, x^i_{t+1})$, is called a risk adjustment or correction. It tells us how the asset's price is affected by risk. An asset whose payoff covaries positively (negatively) with the stochastic discount factor has its priced raised (lowered).

For what follows, we note that assuming the usual characteristics of utility functions stated above, our function will exhibit diminishing marginal utility. Thus, marginal utility $u'(c)$ declines as c increases. Now, let us analyze the effects that some of the variables in (5) will have on prices:

(i) When the expected payoff of asset i, $E(x^i_{t+1})$ is high (low), the price of asset i will be high (low).
(ii) When the risk-free rate is high (low), the price of the asset will be low (high).
(iii) When the asset's payoff x^i_{t+1} covaries positively with future consumption, c_{t+1}, it will covary negatively with the marginal utility of future consumption, $u'(c_{t+1})$, and $\text{cov}(\beta u'(c_{t+1}), x^i) < 0$ thus lowering prices.

Now, let us discuss the intuition behind the above results. **(i)** is clear; it tells us that a high (low) expected return will make the asset more (less) valuable to a trader. **(ii)** tells us that since the trader always has the alternative to invest in the riskless asset that prices of risky assets must decrease (increase) in response to an increase (decrease) in the risk-free rate. Intuitively, when the risk-free rate increases (decrease), demand for the risk free asset will rise (fall) and demand for risky assets fall (rise).

(iii) is more subtle. **(iii)** says if the trader purchases an asset whose payoff covaries positively with his future consumption, one that pays off well when he will be feeling wealthy, and pays off badly when he will be feeling poor, it will make make his consumption stream more volatile. The trader will require a low price to induce him to buy such an asset. If he buys an asset whose payoff covaries negatively with consumption, it helps to smooth consumption and so is more valuable than its expected payoff might indicate.

It is important to note that in the exposition above, the covariance rather than the variance of the payoff with the SDF determined the asset's riskiness. This is because the investor cares about the volatility of his consumption. If he can keep a steady consumption stream, he does not care about the volatility of his portfolio or any individual assets.

4. Systematic & Idiosyncratic Risk, Excess Returns, and Beta Representation

We can decompose any payoff x_{t+1} into a part correlated with the SDF (**systematic** risk) and a part uncorrelated with the sdf (**idiosyncratic** risk) by running the regression

$$x_{t+1} = proj(x_{t+1}|m_{t+1}) + \varepsilon_{t+1} \tag{6}$$

where

$$proj(x_{t+1}|m_{t+1}) := \frac{E(m_{t+1}x_{t+1})}{E(m_{t+1}^2)} m_{t+1} \tag{7}$$

Plugging (7) into (6) and multiplying the resulting equation by m_{t+1}, yields:

$$m_{t+1}x_{t+1} = \frac{E(m_{t+1}x_{t+1})}{E(m_{t+1}^2)} m_{t+1}^2 + m_{t+1}\varepsilon_{t+1}$$

Taking expectations:

$$\begin{aligned} E(m_{t+1}x_{t+1}) &= \frac{E(m_{t+1}x_{t+1})}{E(m_{t+1}^2)} E(m_{t+1}^2) + E(m_{t+1}\varepsilon_{t+1}) \\ &= E(m_{t+1}x_{t+1}) + E(m_{t+1}\varepsilon_{t+1}) \end{aligned}$$

which implies that an asset with payoff ε_{t+1} will be a zero price asset.

$$p_t(\varepsilon_{t+1}) := E(m_{t+1}\varepsilon_{t+1}) = 0$$

In addition, the price of an asset with payoff $proj(x_{t+1}|m_{t+1})$ is

$$\begin{aligned} p_t(proj(x_{t+1}|m_{t+1})) &= p_t\left(\frac{E(m_{t+1}x_{t+1})}{E(m_{t+1}^2)} m_{t+1}\right) \\ &= E\left(\frac{E(m_{t+1}x_{t+1})}{E(m_{t+1}^2)} m_{t+1}^2\right) \\ &= E(m_{t+1}x_{t+1}) = p(x_{t+1}) \end{aligned}$$

Applying the definition of covariance

$$\text{cov}(m_{t+1}, R_{t+1}^i) = E(m_{t+1}R_{t+1}^i) - E(m_{t+1})E(R_{t+1}^i)$$

to (2) we obtain:

$$\begin{aligned} 1 &= E(m_{t+1}R_{t+1}^i) = E(m_{t+1})E(R_{t+1}^i) + \text{cov}(m_{t+1}, R_{t+1}^i) \\ &= E(R_{t+1}^i)/R^f + \text{cov}(m_{t+1}, R_{t+1}^i) \end{aligned} \tag{8}$$

Rearranging and noting that $R^f = 1/E(m_{t+1})$,

$$\begin{aligned} E(R_{t+1}^i) - R^f &= -R^f \underset{t}{\text{cov}}(m_{t+1}, R_{t+1}^i) \\ &= -\frac{\text{cov}(\beta u'(c_{t+1}), R_{t+1}^i)}{E(u'(c_{t+1}))} \end{aligned} \tag{9}$$

- $E(R^i_{t+1}) - R^f$ is defined as the excess return of asset i. It is the return of asset i, net the return of the risk-free asset.
- In words, each assets has a net expected return equal to the risk adjustment.
- Assets whose returns covary positively with consumption make consumption more volatile, and so must promise higher expected returns to induce investors to hold them.
- Assets whose returns covary negatively with consumption, such as insurance, can offer expected rates of return that are lower than the risk-free rate, or even negative (net) expected returns.

We may express (9) as

$$E(R^i_{t+1}) \quad = \quad R^f + \left(\frac{\text{cov}(m_{t+1}, R^i_{t+1})}{\text{var}(m_{t+1})}\right)\left(-\frac{\text{var}(m_{t+1})}{E(m_{t+1})}\right)$$

or

$$E(R^i_{t+1}) \quad = \quad R^f + \beta^{i,m}_t \lambda_m \tag{10}$$

- This is called the **beta pricing model**.
- Note that $\beta^{i,m}_t$ is the regression coefficient of the return R^i on m.
- In words, each expected return should be proportional to the regression coefficient, or beta, in a regression of that return on the SDF m.
- $\beta^{i,m}_t$ is often interpreted as the **quantity of risk**
- λ^m_t is often interpreted as the **price of risk**.

5. Mean-Variance Frontier

In this section we arrive at the thrust of the CCAPM model. We will derive a range in which all assets must lie and determine which assets (or portfolio of assets) will give the highest expected returns for a given amount of variance. To do so, we begin by noting that all assets priced by the sdf m must satisfy

$$|E(R^i_{t+1}) - R^f| \leq \frac{\sigma(m_{t+1})}{E(m_{t+1})}\sigma(R^i_{t+1}) \tag{11}$$

To derive this result, we recall the definition of correlation

$$\rho_{m_{t+1}, R^i_{t+1}} := \frac{\text{cov}\, m_{t+1}, R^i_{t+1}}{\sigma(m_{t+1})\sigma(R^i_{t+1})} \tag{12}$$

Using this definition, from (8) we have:

$$
\begin{aligned}
1 &= E(m_{t+1} R^i_{t+1}) \\
&= E(m_{t+1}) E(R^i_{t+1}) + \mathrm{cov}(m_{t+1}, R^i_{t+1}) \\
&= E(m_{t+1}) E(R^i_{t+1}) + \rho_{m_{t+1}, R^i_{t+1}} \frac{\sigma(m_{t+1})}{E(M_{t+1})}
\end{aligned}
$$

and thus, we get

$$E(R^i_{t+1}) = R^f - \rho_{m_{t+1}, R^i_{t+1}} \frac{\sigma(m_{t+1})}{E(m_{t+1})} \tag{13}$$

Since correlation coefficients are no larger than 1, we obtain (11). From (11) we get many powerful and interesting implications:

I. All means and variances of asset returns will lie in the wedge shaped region illustrated in Figure 1 called the mean-variance region. The boundary of the region is called the mean-variance frontier (MVF). The mean-variance frontier tells us what minimal and maximal mean returns are attainable for a given variance.

II. Since the frontier is generated by taking $|\rho_{m_{t+1}, R^i_{t+1}}| = 1$, this means that all returns on the MVF are perfectly correlated with the SDF. From (13) we see that returns that on the upper (lower) part of the MVF are perfectly negatively (positively) correlated with the SDF and hence positively (negatively) correlated with consumption. That is returns that are on the upper part of the MVF are "‘maximally risky"’ since they maximally positively covary with consumption. Returns that are on the lower part of the MVF provide the best insurance against fluctuations in consumption since they maximally negatively covary with consumption.

III. Since each asset return on the MVF is perfectly correlated with the SDF, for a given frontier return R^{mv}_{t+1} we can pick constants $a, b \in \mathbb{R}$ such that

$$m_{t+1} = a + b R^{mv}_{t+1} \tag{14}$$

Therefore, any frontier return carries all pricing information. That is, given a frontier return and the risk-free rate, we can find a SDF that prices all factors and vice versa. In an appendix post, we will show that the constants a and b can be expressed in terms of the risk-free rate and the expected return of asset mv.

IV. Since each asset return on the MVF is perfectly correlated with the SDF, they are also perfectly correlated with each other. This implies that we can span or synthesize any frontier return from the risk-free return and one other frontier return. That is, given a frontier return R^m then any other frontier return R^{mv} must be expressible as

$$R^{mv}_{t+1} = R^f + a(R^m_{t+1} - R^f) \tag{15}$$

for some $a \in \mathbb{R}$.

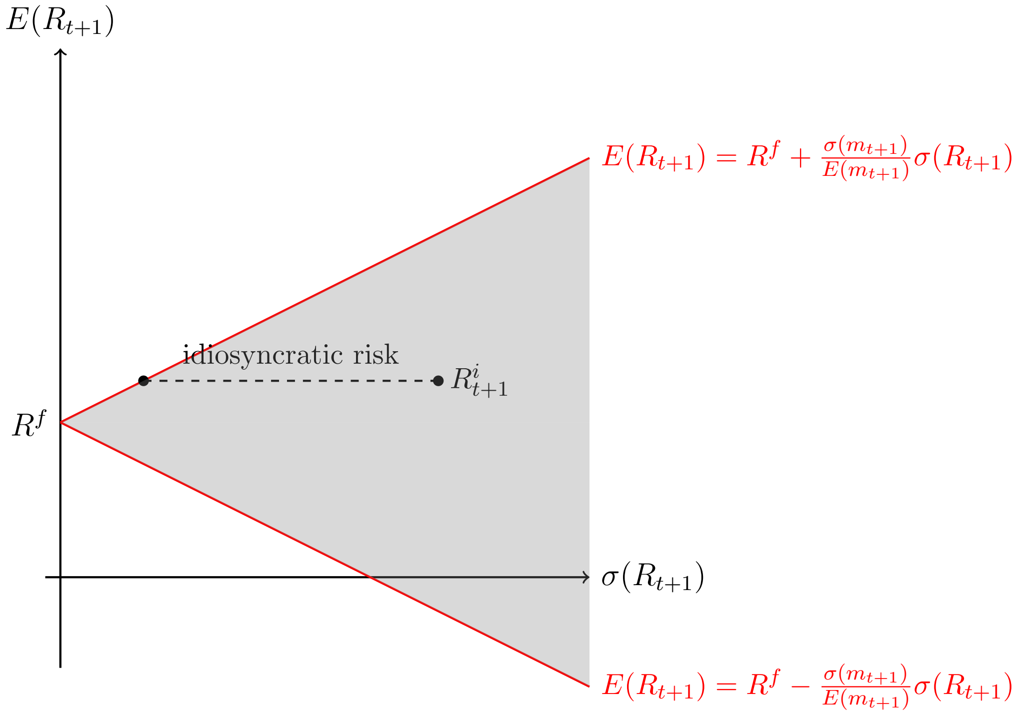

FIGURE 1. Mean-Variance Frontier. All asset returns lie within the shaded wedge or on it's boundary. The boundary is called the mean-variance frontier. The entire wedge region is called the mean-variance region.

V. Given a SDF, we can construct a single-beta representation using any frontier return (except the risk-free rate) so expected returns can be described in the following way:

$$E(R^i_{t+1}) = R^f + \beta_t^{i,mv}[E(R^{mv} - R^f)] \tag{16}$$

where $\beta_t^{i,mv} = \text{cov}(R^i_{t+1}, R^{mv}_{t+1})$. We can identify the factor risk premium or the price of risk as $\lambda = E(R^{mv}_{t+1} - R^f)$ since this beta pricing model applies to every return including R^{mv} and $\beta_t^{mv,mv} = 1$. The essence of (16) is that, even though the means and standard deviations of returns fill out the region inside the MVF, a graph of means versus betas will yield a straight line. We will go through the derivation of (16) in an appendix post.

VI. We can plot the decomposition of a return into a "'priced"' or "systematic" component and a "residual", or "idiosyncratic" component as shown in Figure 1. The systematic component is perfectly correlated with the SDF, and thus perfectly correlated with any frontier return. The idiosyncratic component is uncorrelated with the SDF or any frontier return. In addition, since the idiosyncratic component generates no expected return, it only effects the variance of the asset, and thus it lies horizontal as shown in the figure.